



6.6 Spans and subspaces

Douglas Wilhelm Harder, LEL, M.Math.

dwharder@uwaterloo.ca

dwharder@gmail.com





Introduction

- In this topic, we will
 - Review the definition of a span
 - Discuss when we can and cannot remove a vector from a collection and leave the span unchanged
 - See an algorithm for minimizing the number of vectors in a collection while leaving the span unchanged
 - Determine when the spans of two collections of vectors are equal





Definition

- Recall the definition of the span of a collection of vectors:

Definition

The span of a collection of vectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ is the set of all linear combinations of these n vectors.

- We denote the span of these vectors by $\text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$
 - Note that $\text{span}\{\} = \{\mathbf{0}\}$
 - Recall that the sum of no numbers being 0





Membership in the span

- If a vector $\mathbf{v}$ is in the span of a collection of vectors,
we will write $\mathbf{v} \in \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$
 - This is read as “ $\mathbf{v}$ is in the span of $\mathbf{u}_1$ through $\mathbf{u}_n$ ”
or “ $\mathbf{v}$ is an element of the span of $\mathbf{u}_1$ through $\mathbf{u}_n$ ”
- If a vector $\mathbf{v}$ is not in the span of a collection of vectors,
we will write $\mathbf{v} \notin \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$
 - This is read as “ $\mathbf{v}$ is not in the span of $\mathbf{u}_1$ through $\mathbf{u}_n$ ”
or “ $\mathbf{v}$ is not an element of the span of $\mathbf{u}_1$ through $\mathbf{u}_n$ ”





Relationships

- Note that $\mathbf{v} \in \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$,
if and only if there exists at one solution to

$$\alpha_1 \mathbf{u}_1 + \dots + \alpha_n \mathbf{u}_n = \mathbf{v}$$

if and only if $\mathbf{v}$ depends on $\mathbf{u}_1, \dots, \mathbf{u}_n$

- Note that $\mathbf{0}$ is in every span,
as there is always at least one solution to

$$\alpha_1 \mathbf{u}_1 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

- Thus, $\mathbf{0}$ depends on any collection of vectors $\mathbf{u}_1, \dots, \mathbf{u}_n$,
including none





Observations

- The span of any collection of vectors is a subspace of the vector space from which the vectors are in
- In $\mathbf{F}^n$, every subspace is the span of a finite collection of vectors
 - We will prove this in time
- In other vector spaces, this is not necessarily the case
 - Consider all bounded discrete-time signals





Equivalent spans

Theorem

If there is a vector $\mathbf{u}_k$ in the span $\text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ such that

$$\mathbf{u}_k \in \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$$

then $\text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\} = \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$

Proof: If $\mathbf{v} \in \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ and $\mathbf{u}_k \in \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ then $\mathbf{v} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_n \mathbf{u}_n$

$$\mathbf{u}_k = \beta_1 \mathbf{u}_1 + \dots + \beta_{k-1} \mathbf{u}_{k-1} + \beta_{k+1} \mathbf{u}_{k+1} + \dots + \beta_n \mathbf{u}_n$$

Substituting the second into the first yields:

$$\mathbf{v} = \alpha_1 \mathbf{u}_1 +$$

$$\begin{aligned} &= (\alpha_1 + \alpha_k \beta_1) \mathbf{u}_1 + \dots + (\alpha_{k-1} + \alpha_k \beta_{k-1}) \mathbf{u}_{k-1} \\ &\quad + (\alpha_{k+1} + \alpha_k \beta_{k+1}) \mathbf{u}_{k+1} + \dots + (\alpha_n + \alpha_k \beta_n) \mathbf{u}_n \end{aligned}$$

Thus, $\mathbf{v} \in \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ ■





Equivalent spans

Theorem

Given a collection of vectors $\mathbf{u}_1, \dots, \mathbf{u}_n$, if there is a vector $\mathbf{u}_k$ that is linearly independent of the other vectors, then $\text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is a strict subspace of $\text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$.

Proof: First, any span of a collection of vectors is a subspace.

Also, every vector in $\text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is also in $\text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$

After all, if $\mathbf{v} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_{k-1} \mathbf{u}_{k-1} + \alpha_{k+1} \mathbf{u}_{k+1} + \dots + \alpha_n \mathbf{u}_n$
 $= \alpha_1 \mathbf{u}_1 + \dots + \alpha_{k-1} \mathbf{u}_{k-1} + 0 \cdot \mathbf{u}_k + \alpha_{k+1} \mathbf{u}_{k+1} + \dots + \alpha_n \mathbf{u}_n$

However, $\mathbf{u}_k$ is linearly independent of the other vectors,

so by definition, $\mathbf{u}_k \notin \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$

Thus, the span of the smaller collection is a strict subspace of the span of the larger collection. ■





Example

- Thus, if any vector in $\mathbf{u}_1, \dots, \mathbf{u}_n$ is dependent on the others, we may remove it from the collection and the span is unchanged
- Also, if any vector is independent of the others, we cannot remove it from the collection without changing the span
- Consider the span of these five vectors:

$$\text{span} \left\{ \begin{pmatrix} 2 \\ 1.6 \\ -0.6 \end{pmatrix}, \begin{pmatrix} 4 \\ 3.2 \\ -1.2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2.2 \\ 5.3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1.2 \\ -2.9 \end{pmatrix}, \begin{pmatrix} 5 \\ 5.8 \\ 1.5 \end{pmatrix} \right\}$$

- This collection of vectors is linearly dependent:
 - We have five vectors in $\mathbf{R}^3$





Example

- From before, we saw that:

$$\begin{pmatrix} 2 & 4 & -1 & 3 & 5 \\ 1.6 & 3.2 & 2.2 & 1.2 & 5.8 \\ -0.6 & -1.2 & 5.3 & -2.9 & 1.5 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 & 3 & 5 \\ 0 & 0 & 5 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Thus, the free variables are associated with Columns 2, 4 and 5

- Thus, $\begin{pmatrix} 2 & \vdots & 4 \\ 1.6 & \vdots & 3.2 \\ -0.6 & \vdots & -1.2 \end{pmatrix} \sim \begin{pmatrix} 2 & \vdots & 4 \\ 0 & \vdots & 0 \\ 0 & \vdots & 0 \end{pmatrix}$

- Thus, we note

$$\begin{pmatrix} 4 \\ 3.2 \\ -1.2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 2 \\ 1.6 \\ -0.6 \end{pmatrix} + 0 \cdot \begin{pmatrix} -1 \\ 2.2 \\ 5.3 \end{pmatrix} + 0 \cdot \begin{pmatrix} 3 \\ 1.2 \\ -2.9 \end{pmatrix} + 0 \cdot \begin{pmatrix} 5 \\ 5.8 \\ 1.5 \end{pmatrix}$$





Example

- Thus,

$$\text{span} \left\{ \begin{pmatrix} 2 \\ 1.6 \\ -0.6 \end{pmatrix}, \begin{pmatrix} 4 \\ 3.2 \\ -1.2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2.2 \\ 5.3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1.2 \\ -2.9 \end{pmatrix}, \begin{pmatrix} 5 \\ 5.8 \\ 1.5 \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} 2 \\ 1.6 \\ -0.6 \end{pmatrix}, \begin{pmatrix} -1 \\ 2.2 \\ 5.3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1.2 \\ -2.9 \end{pmatrix}, \begin{pmatrix} 5 \\ 5.8 \\ 1.5 \end{pmatrix} \right\}$$





Example

- Again,

$$\begin{pmatrix} 2 & 4 & -1 & 3 & 5 \\ 1.6 & 3.2 & 2.2 & 1.2 & 5.8 \\ -0.6 & -1.2 & 5.3 & -2.9 & 1.5 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 & 3 & 5 \\ 0 & 0 & 5 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Taking just Columns 1, 3 and 4:

$$\begin{pmatrix} 2 & -1 & \vdots & 3 \\ 1.6 & 2.2 & \vdots & 1.2 \\ -0.6 & 5.3 & \vdots & -2.9 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & \vdots & 3 \\ 0 & 5 & \vdots & -2 \\ 0 & 0 & \vdots & 0 \end{pmatrix}$$

- Thus, we note

$$\begin{pmatrix} 3 \\ 1.2 \\ -2.9 \end{pmatrix} = 1.3 \cdot \begin{pmatrix} 2 \\ 1.6 \\ -0.6 \end{pmatrix} - 0.4 \cdot \begin{pmatrix} -1 \\ 2.2 \\ 5.3 \end{pmatrix} + 0 \cdot \begin{pmatrix} 5 \\ 5.8 \\ 1.5 \end{pmatrix}$$





Example

- Thus,

$$\text{span} \left\{ \begin{pmatrix} 2 \\ 1.6 \\ -0.6 \end{pmatrix}, \begin{pmatrix} 4 \\ 3.2 \\ -1.2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2.2 \\ 5.3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1.2 \\ -2.9 \end{pmatrix}, \begin{pmatrix} 5 \\ 5.8 \\ 1.5 \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} 2 \\ 1.6 \\ -0.6 \end{pmatrix}, \begin{pmatrix} -1 \\ 2.2 \\ 5.3 \end{pmatrix}, \begin{pmatrix} 5 \\ 5.8 \\ 1.5 \end{pmatrix} \right\}$$





Example

- Finally,

$$\begin{pmatrix} 2 & 4 & -1 & 3 & 5 \\ 1.6 & 3.2 & 2.2 & 1.2 & 5.8 \\ -0.6 & -1.2 & 5.3 & -2.9 & 1.5 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 & 3 & 5 \\ 0 & 0 & 5 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Taking just Columns 1, 3 and 5:

$$\begin{pmatrix} 2 & -1 & 5 \\ 1.6 & 2.2 & 5.8 \\ -0.6 & 5.3 & 1.5 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 5 \\ 0 & 5 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

- Thus, we note

$$\begin{pmatrix} 5 \\ 5.8 \\ 1.5 \end{pmatrix} = 2.8 \cdot \begin{pmatrix} 2 \\ 1.6 \\ -0.6 \end{pmatrix} + 0.6 \cdot \begin{pmatrix} -1 \\ 2.2 \\ 5.3 \end{pmatrix}$$





Example

- Finally, we see that:

$$\text{span} \left\{ \begin{pmatrix} 2 \\ 1.6 \\ -0.6 \end{pmatrix}, \begin{pmatrix} 4 \\ 3.2 \\ -1.2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2.2 \\ 5.3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1.2 \\ -2.9 \end{pmatrix}, \begin{pmatrix} 5 \\ 5.8 \\ 1.5 \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} 2 \\ 1.6 \\ -0.6 \end{pmatrix}, \begin{pmatrix} -1 \\ 2.2 \\ 5.3 \end{pmatrix} \right\}$$

- Note that from

$$\begin{pmatrix} 2 & 4 & -1 & 3 & 5 \\ 1.6 & 3.2 & 2.2 & 1.2 & 5.8 \\ -0.6 & -1.2 & 5.3 & -2.9 & 1.5 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 & 3 & 5 \\ 0 & 0 & 5 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

we get that

$$\begin{pmatrix} 2 & -1 \\ 1.6 & 2.2 \\ -0.6 & 5.3 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 \\ 0 & 5 \\ 0 & 0 \end{pmatrix}$$

- Thus, the two remaining vectors are linearly independent





Example

- We can also proceed this way:

- Given

$$\begin{pmatrix} 2 \\ 1.6 \\ -0.6 \end{pmatrix}, \begin{pmatrix} 4 \\ 3.2 \\ -1.2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2.2 \\ 5.3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1.2 \\ -2.9 \end{pmatrix}, \begin{pmatrix} 5 \\ 5.8 \\ 1.5 \end{pmatrix}$$

- We note that Columns 2, 4 and 5 are associated with free variables

$$\begin{pmatrix} 2 & 4 & -1 & 3 & 5 \\ 1.6 & 3.2 & 2.2 & 1.2 & 5.8 \\ -0.6 & -1.2 & 5.3 & -2.9 & 1.5 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 & 3 & 5 \\ 0 & 0 & 5 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Thus, strike those vectors out

$$\text{span} \left\{ \begin{pmatrix} 2 \\ 1.6 \\ -0.6 \end{pmatrix}, \begin{pmatrix} 4 \\ 3.2 \\ -1.2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2.2 \\ 5.3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1.2 \\ -2.9 \end{pmatrix}, \begin{pmatrix} 5 \\ 5.8 \\ 1.5 \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} 2 \\ 1.6 \\ -0.6 \end{pmatrix}, \begin{pmatrix} -1 \\ 2.2 \\ 5.3 \end{pmatrix} \right\}$$

- The remaining vectors are linearly independent





Testing for equality

- Suppose we have the spans of two collections of vectors:

$$\text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_m\} \text{ and } \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$$

how do you know if these spans are equal?

- If you know nothing else about these collections, you must test:
 - If $\mathbf{v}_k \in \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ for all $k = 1, \dots, n$
 - If $\mathbf{u}_k \in \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for all $k = 1, \dots, m$
- That is, you must attempt to solve

$$\mathbf{v}_k = \alpha_1 \mathbf{u}_1 + \dots + \alpha_m \mathbf{u}_m$$

for all $k = 1, \dots, n$ and you must attempt to solve

$$\mathbf{u}_k = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n$$

for all $k = 1, \dots, m$

- As soon as one has no solutions, you may stop,
the spans are not equal





Summary

- Following this topic, you now
 - Have a reinforced understanding of the span of a collection of vectors
 - Know that membership of a vector in a span is equivalent to dependence of that vector on the collection, which is equivalent to being able to solve for that vector as a linear combination of the collection
 - Given the span of a collection,
 - if one vector linearly depends on the others,
 - we can remove it and the span is unchanged
 - if one vector is linearly independent of the others
 - we cannot remove it without changing the span
 - Determining if the spans of two collections are equal requires us to check if each vector is in the span of the other collection





References

- [1] https://en.wikipedia.org/wiki/Linear_independence
- [2] https://en.wikipedia.org/wiki/Linear_span





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

<https://www.rbg.ca/>

for more information.





Disclaimer

These slides are provided for the NE 112 *Linear algebra for nanotechnology engineering* course taught at the University of Waterloo. The material in it reflects the authors' best judgment in light of the information available to them at the time of preparation. Any reliance on these course slides by any party for any other purpose are the responsibility of such parties. The authors accept no responsibility for damages, if any, suffered by any party as a result of decisions made or actions based on these course slides for any other purpose than that for which it was intended.

